ON CONTINUOUS FIELDS OF JB-ALGEBRAS

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Dedicated to the memory of Professor George Bachman, Polytechnic University, Brooklyn, NY, USA.

Abstract. We introduce and study continuous fields of JB-algebras (which are real non-associate analogues of C*-algebras). In particular, we show that for the universal enveloping C*-algebra $C_u^*(B)$ for the JB-algebra B defined by a continuous field of JB-algebras A_t, $t \in T$, on a locally compact space T there exists a decomposition of $C_u^*(B)$ into a continuous field of C*-algebras $C_u^*(A_t)$, $t \in T$, on the same space T, composed entirely of the universal enveloping C*-algebras of the corresponding JB-algebras from the aforementioned decomposition of the algebra B.

1. Introduction and Preliminaries

Banach associative regular *-algebras over $\mathbb{C}$, so called *C*-algebras*, were first introduces by Gelfand and Naimark in the paper [5]. Since then these algebras were studied extensively by various authors. This theory now is a big subdomain of the of Functional Analysis as a subject which found applications in almost all branches of Modern Mathematics and Physics. For the basics of the theory of C*-algebras, see for example Pedersen's monograph [9]. The basic theory of real associative analogues of C*-algebras, so called *real C*-algebras*, is presented in Li's monograph [8].

In order to obtain a topological non-commutative version of Gelfand's characterization of commutative C*-algebras, Dixmier and Douady in [4] introduced a notion of continuous fields of Banach spaces and C*-algebras, which found important applications in classification of C*-algebras (see [3]) and Theoretical Physics (see [7]). According to them, **a continuous field of C*-algebras**

$$(\mathfrak{B}, \{\mathfrak{A}_t, \varphi_t\}_{t\in T}),$$

over a locally compact Hausdorff space T consists of a C-algebra $\mathfrak{B}$, a collection of C*-algebras*

$$\{\mathfrak{A}_t\}_{t\in T},$$

and a set

$$\{\varphi_t : \mathfrak{B} \to \mathfrak{A}_t\}_{t\in T},$$

of surjective morphisms, such that:

1). The function

$$t \mapsto \|\varphi_t(x)\|,$$

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is in $C_0(T)$ *for all* $x \in \mathfrak{B}$;

2). The norm of any $x \in \mathfrak{B}$ *is*

$$\|x\|_{\mathfrak{B}} = \sup_{t\in T} \|\varphi_t(x)\|_{\mathfrak{A}_t};$$

3). For any $f \in C_0(T)$ *and* $x \in \mathfrak{B}$, *there is an element*

$$fx \in \mathfrak{B},$$

for which

$$\varphi_t(fx) = f(t)\varphi_t(x),$$

for all $t \in T$.

A ***section*** *of the field is an element* $\{x_t\}_{t\in T}$ *of*

$$\prod_{t\in T} \mathfrak{A}_t,$$

for which there is an element $x \in \mathfrak{B}$ *such that*

$$x_t = \varphi_t(x),$$

for all $t \in T$.

One can see that $\mathfrak{B}$ *can be identified with the space of sections of the field, seen as a C*-algebra under pointwise scalar multiplication, addition, adjointing, and operator multiplication, by means*

$$\{\varphi_t(x)\}_{t\in T} \leftrightarrow x.$$

In particular,

$$x = y,$$

iff

$$\varphi_t(x) = \varphi_t(y),$$

for all t. *It is natural that algebra* $\mathfrak{B}$ *is called a the* ***C*-algebra of the continuous field of C*-algebras****.*

Since the beginning of the theory of complex C*-algebras, there were numerous attempts to extend this theory to non-associative algebras which are close to associative, in particular to Jordan algebras. In fact, Alfsen, Shultz and Størmer in [2] defined so called *JB-algebras* as the real Banach–Jordan algebras satisfying for all pairs of elements x and y the inequality of fineness

$$\left\|x^2 + y^2\right\| \geq \|x\|^2,$$

and regularity condition

$$\left\|x^2\right\| = \|x\|^2.$$

If $\mathfrak{A}$ is a C*–algebra, or a real C*-algebra, then the self-adjoint part $\mathfrak{A}_{sa}$ of $\mathfrak{A}$ is a JB-algebra under the Jordan product

$$x \circ y = \frac{(xy + yx)}{2}.$$

Closed subalgebras of $\mathfrak{A}_{sa}$, for some C*-algebra or real C*-algebra $\mathfrak{A}$, become relevant examples of JB-algebras, and are called *JC-algebras*.

The basic theory of JB-algebras is fully treated in monograph of Hanche-Olsen and Størmer [6].In particular, in this monograph there is the following theorem which was for the first time presented by Alfsen, Hanche-Olsen and Shultz in the paper [1].

Theorem 1 (Alfsen, Hanche-Olsen, Shultz [1]). *For an arbitrary JB-algebra A there exists a unique up to an isometric *-isomorphism a C^*-algebra $C_u^*(A)$ (**the universal enveloping C*-algebra for the JB-algebra A**), and a Jordan homomorphism*

$$\psi_A : A \to C_u^*(A)_{sa},$$

from A to the self-adjoint part of $C_u^*(A)$, such that:

1). $\psi_A(A)$ generates $C_u^*(A)$ as a C^*-algebra;

2). for any pair composed of a C^*-algebra $\mathfrak{A}$ and a Jordan homomorphism

$$\rho : A \to \mathfrak{A}_{sa},$$

*from A into the self-adjoint part of $\mathfrak{A}$, there exists a *-homomorphism*

$$\widehat{\rho} : C_u^*(A) \to \mathfrak{A},$$

from the C^*-algebra $C_u^*(A)$ into C^*-algebra $\mathfrak{A}$, such that

$$\rho = \widehat{\rho} \circ \psi_A;$$

*3). there exists a *-antiautomorphism Φ of order 2 on the C^*-algebra $C_u^*(A)$, such that*

$$\Phi(\psi_A(a)) = \psi_A(a),$$

$\forall a \in A$. □

Our plan is to define a *continuous field of JB-algebras*, the *JB-algebra of the continuous field of JB-algebras*, and be able in the spirit of Theorem 1 above to associate in a universal sense with each JB-algebra of the continuous field of JB-algebras a C*-algebra of the continuous field of C*-algebras.

2. Continuous fields of JB-algebras

Let us first introduce a continuous field of JB-algebras.

Definition 1. *A **continuous field of JB-algebras***

$$(B, \{A_t, \varphi_t\}_{t \in T}),$$

over a locally compact Hausdorff space T consists of a JB-algebra B, a collection of JB-algebras $\{A_t\}_{t \in T}$,and a set

$$\{\varphi_t : B \to A_t\}_{t \in T},$$

of surjective morphisms, such that:

1). The function

$$t \mapsto \|\varphi_t(x)\|,$$

is in $C_0(T)$ for all $x \in B$;

2). The norm of any $x \in B$ is

$$\|x\|_B = \sup_{t \in T} \|\varphi_t(x)\|_{A_t};$$

3). For any $f \in C_0(T)$ and $x \in B$, there is an element

$$fx \in B,$$

for which

$$\varphi_t(fx) = f(t)\varphi_t(x),$$

for all $t \in T$.

A ***section*** *of the field is an element* $\{x_t\}_{t\in T}$ *of*

$$\prod_{t\in T} A_t,$$

for which there is an element $x \in B$ *such that*

$$x_t = \varphi_t(x),$$

for all $t \in T$.

We identify B *with the space of sections of the field, seen as a JB-algebra under pointwise scalar multiplication, addition, operator multiplication, by means*

$$\{\varphi_t(x)\}_{t\in T} \leftrightarrow x.$$

In particular,

$$x = y,$$

iff

$$\varphi_t(x) = \varphi_t(y),$$

for all t. *It is natural that algebra* B *is called a the* ***JB-algebra of the continuous field of JB-algebras****.*

Now we will establish a few properties of the continuous field of JB-algebras. The first one is about locally uniform closedness of the continuous field of JB-algebras.

Proposition 1. *The JB-algebra* B *of sections of a continuous field of JB-algebras is* ***locally uniformly closed****, i.e. if*

$$x \in \prod_{t\in T} A_t,$$

is such that for every

$$s \in T,$$

and every $\varepsilon > 0$ *there exists*

$$y_s \in B,$$

and a neighborhood

$$V_s \subset T,$$

of s *in which*

$$\|x_t - \varphi_t(y_s)\| < \varepsilon,$$

for all

$$t \in V_s,$$

and also

$$\lim_{t\to\infty} \|x_t\| = 0,$$

then

$$x \in B.$$

Alternatively, if the function

$$t \mapsto \|x_t - z_t\|,$$

lies in $C_0(T)$ *for each*

$$z \in B,$$

then

$$x \in B.$$

Proof. Under conditions of the first part of the Proposition, there exists a compact set

$$K \subseteq T,$$

for which

$$\|x_t\| < \varepsilon,$$

outside of K, as well as a finite cover

$$\{V_{t_1}, ..., V_{t_n}\},$$

of K,

$$K \subseteq \{V_{t_1}, ..., V_{t_n}\}.$$

Now we have to recall a notion of a partition of unity on K subordinate to this cover (see for example [10] and [7]). Let K be a Hausdorff space, and

$$\{V_\alpha\}_{\alpha \in \Lambda},$$

be a **locally finite open cover** of K, i.e. each point of of K has a neighborhood that intersects only with a finite number of the sets V_α. A **partition of unity subordinate to the given cover** is a collection of positive functions

$$\{u_\alpha\}_{\alpha \in \Lambda},$$

such that each u_α is a compactly supported continuous real-valued function with

$$\sum_{\alpha \in \Lambda} u_\alpha = 1.$$

A partition of unity always exists when K is paracompact (see [10]). So, let us take a partition of unity

$$\{u_i\}_{i=1}^n,$$

on K subordinate to the aforementioned finite cover

$$\{V_{t_1}, ..., V_{t_n}\}.$$

Let us consider the

$$y = \sum_{i=1}^n u_i y_{t_i}.$$

From Definition 1.3 it follows that

$$y \in B,$$

and satisfies the condition

$$\sup_{t \in T} \|x_t - y_t\| < \varepsilon.$$

Therefore, from Definition 1.2 and completeness of B it follows that

$$x \in B.$$

Now, given any

$$x \in \prod_{t \in T} A_t,$$

and

$$s \in T,$$

because φ_s is surjective, there exists an element

$$y_s \in B,$$

such that
$$x_s = \varphi_s(y_s).$$
The assumption of the second part of Proposition 1 then implies that the conditions in the first part of this Proposition are satisfied, such that
$$x \in B.$$
□

The following Proposition gives conditions for the existence and uniqueness of a continuous field of JB-algebras whose collection of sections contains a subset possessing some natural properties.

Proposition 2. *Let*
$$\{A_t\}_{t\in T},$$
be a family of JB-algebras indexed by a locally compact Hausdorff space T, and a subset
$$\widetilde{B} \subseteq \prod_{t\in T} A_t,$$
that satisfies the following properties:

1). The set
$$\{x_t : x \in \widetilde{B}\},$$
is dense in A_t for each $t \in T$;

2). The function
$$t \mapsto \|x_t\|,$$
lies in $C_0(T)$ for each $x \in \widetilde{B}$;

3). The set $\widetilde{B}$ is a Jordan algebra under pointwise operations.

Then there exists a unique continuous field of JB-algebras
$$(B, \{A_t, \varphi_t\}_{t\in T}),$$
whose collection of sections contains $\widetilde{B}$. Namely, B consists of all
$$x \in \prod_{t\in T} A_t,$$
for which the function
$$x \mapsto \|x_t - z_t\|,$$
lies in $C_0(T)$ for each $z \in \widetilde{B}$, regarded as JB-algebra under pointwise operations, and the norm of Definition 1.2. Finally,
$$\varphi_t(x) = x_t,$$
$t \in T$, is the evaluation map.

Proof. We show first that the algebra B defined above is locally uniformly closed. With the objects x, s, ε, y_s and V as specified in Proposition 1, take $z \in \widetilde{B}$ arbitrary, and define the functions
$$f_{xz} : t \mapsto \|x_t - z_t\|,$$
and
$$f_{yz} : t \mapsto \|\varphi_t(y_s) - z_t\|.$$
Using the triangle inequality for the norm in Banach space, we get that
$$|(\|x\| - \|y\|)| \leq \|x - y\|,$$

and that gives us that

$$|f_{xz}(t) - f_{yz}(t)| < \varepsilon,$$

for all $t \in V$. By assumption, the function f_{yz} is continuous, so that

$$|f_{yz}(t) - f_{yz}(s)| < \varepsilon,$$

for all t's in some neighborhood V' of s. Combining the inequalities, we get

$$|f_{xz}(t) - f_{xz}(s)| < 3\varepsilon,$$

for all

$$t \in V \cap V'.$$

Therefore f_{xz} is continuous at s, which was arbitrary, so that $x \in B$ by the definition of B.

Now, we show uniqueness of B. Using this property one can easily see that B is a JB-algebra, and that the condition 3 in Definition 1 is satisfied. It is clear from Definition 1.1 and the definition of B in Proposition 2 that B is maximal. On the other hand, according to the second part of Proposition 1, B is minimal, so, B is unique. □

We are ready now to present the main result of the paper.

Definition 2. *An *-isomorphism (resp. Jordan isomorphism) of continuous fields of C*-algebras (resp. JB-algebras) over the same base Hausdorff locally compact topological space T is the isometric *-isomorphism (Jordan isometric isomorphism) of the C*-algebras (resp. JB-algebras) of the continuous fields via a map respecting the fibers.*

Theorem 2. *For an arbitrary continuous field of JB-algebras*

$$(B, \{A_t, \varphi_t\}_{t\in T}),$$

*over a locally compact Hausdorff space T, there exists a unique up to an *-isomorphism a continuous field of C*-algebras*

$$(C_u^*(B), \{C_u^*(A_t), \widehat{\varphi}_t\}_{t\in T}),$$

(the universal enveloping continuous field of C*-algebras for the continuous field of JB-algebras

$$(B, \{A_t, \varphi_t\}_{t\in T})$$

***over the same base space** T), and a Jordan homomorphism*

$$\psi_B : B \to C_u^*(B)_{sa},$$

from B to the self-adjoint part of $C_u^*(B)$, as well as a family of Jordan homomorphisms

$$\psi_{A_t} : A_t \to C_u^*(A_t)_{sa},$$

$t \in T$, from A_t to the self-adjoint part of $C_u^*(A_t)$, for each $t \in T$, such that:

1). $\psi_B(B)$ generates $C_u^*(B)$ as a C-algebra, and each $\psi_{A_t}(B)$ generates each $C_u^*(A_t)$ as a C*-algebra for each $t \in T$;*

2). for any pair composed of a continuous field of C-algebras $(\mathfrak{B}, \{\mathfrak{A}_t, \widetilde{\varphi}_t\}_{t\in T})$ and a family of Jordan homomorphisms*

$$\rho : B \to \mathfrak{B}_{sa},$$

from B into the self-adjoint part of $\mathfrak{B}$, and

$$\rho_t : A_t \to (\mathfrak{A}_t)_{sa},$$

*for each $t \in T$, from A_t into the self-adjoint part of $\mathfrak{A}_t$, there exist a *-homomorphism*

$$\widehat{\rho} : C_u^*(B) \to \mathfrak{B},$$

*from the C^*-algebra $C_u^*(B)$ into C^*-algebra $\mathfrak{B}$, and a family of *-homomorphisms*

$$\widehat{\rho}_t : C_u^*(A_t) \to \mathfrak{A}_t,$$

for each $t \in T$, from the C^*-algebra $C_u^*(A_t)$ into C^*-algebra $\mathfrak{A}_t$ such that

$$\rho = \widehat{\rho} \circ \psi_B,$$

and

$$\rho_t = \widehat{\rho}_t \circ \psi_{A_t},$$

for each $t \in T$;

*3). there exists a *-antiautomorphism Φ of order 2 on the C^*-algebra $C_u^*(B)$, such that*

$$\Phi(\psi_B(x)) = \psi_B(x),$$

*$\forall x \in B$, as well as there exists a family of *-antiautomorphism Φ_t of order 2 on the C^*-algebra $C_u^*(A_t)$, for each $t \in T$, such that*

$$\Phi(\psi_{A_t}(x_t)) = \psi_{A_t}(x_t),$$

$\forall x_t \in A_t$, and every $t \in T$.

Proof. Let $(B, \{A_t, \varphi_t\}_{t\in T})$ be a given continuous field of JB-algebras. Let $C_u^*(B)$ be the universal enveloping C*-algebra for the JB-algebra of the continuous field, and the family of C*-algebras $C_u^*(A_t)$ for each $t \in T$ be the universal enveloping C*-algebra for the JB-algebra A_t, $t \in T$. Let

$$\psi_B : B \to C_u^*(B)_{sa},$$

and for each $t \in T$,

$$\varphi_t : B \to A_t,$$

and

$$\psi_{A_t} : A_t \to C_u^*(A_t)_{sa}.$$

From Theorem 1 it follows that $\psi_B(B)$ is dense in $C_u^*(B)_{sa}$, and $\psi_{A_t}(A_t)$ is dense in $C_u^*(A_t)_{sa}$ for each $t \in T$. So, without a loss of generality we can assume that for each

$$x, y \in C_u^*(B)_{sa}$$

there exist $a_n, b_n \in B$, such that

$$x = \lim_{n\to\infty} \psi_B(a_n),$$

and

$$y = \lim_{n\to\infty} \psi_B(b_n),$$

where the limit is taken in the norm of $C_u^*(B)$, as well as

$$x_t = \widehat{\varphi}_t(x) = \lim_{n\to\infty} \psi_{A_t}(\varphi_t(a_n)),$$

and

$$y_t = \widehat{\varphi}_t(y) = \lim_{n\to\infty} \psi_{A_t}(\varphi_t(b_n)),$$

$t \in T$, where the limit is taken in the norm of $C_u^*(A_t)$. For each $t \in T$, we will define

$$\widehat{\varphi}_t : C_u^*(B) \to C_u^*(A_t),$$

the following way:

$$\widehat{\varphi}_t(x + iy) = x_t + iy_t.$$

Because $C_u^*(A_t)_{sa}$ is norm closed for each $t \in T$ (see [9]), the last identity it well defined. Moreover, from the fact that φ_t is surjective for each $t \in T$ it follows that $\widehat{\varphi}_t$ is surjective for each $t \in T$ as well. Thus,

$$(C_u^*(B), \{C_u^*(A_t), \widehat{\varphi}_t\}_{t \in T}),$$

is in fact a continuous field of C*-algebras. The rest of the Theorem is obtained as a corollary by application of Theorem 1 in fibers, and Propositions 1 and 2. □

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